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A PRODUCTION SMOOTHING PROBLEM

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√ P-610

6 January 1955
Rev. 28 February 1955

Approved for OTS release

SUMMARY

A single item is to be produced over a given number of time periods to satisfy known future requirements while minimizing costs where the costs per unit for production, storage, and change in production rate are known functions of time.

While such a problem can be solved by regular linear programming methods, the novel feature here is that the primal and dual problems are solved jointly by means of a rapid graphical method involving only intersections and rotations of straight lines. The underlying reason for this stems from a special property of the near "square block triangular" nature of the coefficient matrix. In some cases the answer is given on the first application. Usually only a few iterations are required.

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1. INTRODUCTION

Linear programming problems can be solved by the standard simplex method [1]. For dynamic linear programming problems _formal application_ of this technique is often impractical due to the large number of columns and rows in the matrix of coefficients. There is therefore a need to develop special methods for solving such problems by either a simple explicit formula or at least one involving rapid convergence to the optimal solution. In some problems such as the one discussed in this paper, the special structure of the matrix and the relations between the primal problem and its dual can be exploited to advantage.

The general problem under consideration is similar to one studied by Jacobs and Hoffman; it is to determine a production schedule for a single type item over a number of time periods to satisfy a given requirement schedule while minimizing total cost. Unit production costs, storage costs, and costs of changing production rates are given functions of time.

A special case is that in which production is required to expand or at least be non-decreasing. It will be shown that this expanding production problem can be solved directly in an important case (a convex requirement schedule) and a rapid iterative process is outlined for the general requirement schedule which sometimes gives the answer directly. The same results hold for the more general problem.

The computational simplicity of the proposed method results from two properties of the system:

(1) The primal problem and its dual can be formulated geometrically in the same way.

(2) The values of the basic set of variables can be calculated directly by successively solving one equation for one unknown at each step. Geometrically, this leads immediately to a ruler construction alone.

2. THE EXPANDING PRODUCTION PROBLEM

Let T = the number of periods. $(t = 0, 1, \ldots, T)$

r_t = given requirement due at time t.

x_t = unknown amount produced from time $t-1$ to t.

x_0 = a given constant.

$y_t = x_{t+1} - x_t \geq 0$ = increase in production rate at time t.

To express the condition that the requirement schedule is satisfied for $i = 1 \ldots T$ let

$$X_i = \sum_{t=1}^{i} x_t = \text{the total production from } t = 0 \text{ to } t = i ,$$

$$R_i = \sum_{t=1}^{i} r_t = \text{the total requirement from } t = 0 \text{ to } t = i ,$$

$$R_0 = 0 .$$

Then the excess of accumulated production over accumulated requirements up to time i is given by

$$u_i = u_0 + X_i - R_i \geq 0$$

where u_0 is a given constant, the excess production at the start. To express the costs let

c_i = cost of producing each unit in the period from $i - 1$ to i ,

d_1 = cost of storing each unit of excess u_1 for one period.

e_1 = cost of increasing production rate one unit per unit time at time 1.

Then the problem is to minimize the total cos's, i.e.,

(1) minimize $\sum_{1=0}^{T} (c_1 x_1 + d_1 u_1 + e_1 y_1)$

subject to

(2) $\qquad \sum_{t=1}^{1} x_t \geq \sum_{t=1}^{1} r_t \qquad\qquad 1 = 1 \ldots T$

and $\qquad\qquad x_1, y_1, u_1 \geq 0, \qquad\qquad 1 = 0 \ldots T$.

This problem can be restated in terms of an equivalent system in y_t and u_t alone. Introduce the constants

$$R_1 = e_1 + \sum_{t=1+1}^{T} c_t + \sum_{t=1}^{T-1} (t-1)d_t , \quad 1=0,\ldots,T-1$$

and let $\qquad F_T = 0$; these represent the potential increase in total cost due to increasing the production rate by one unit at time 1. Note that

$$x_t = \sum_{1=1}^{t} x_1 = \sum_{1=0}^{t-1} (t-1)y_1 + t x_0$$

so that

$$u_t = \sum_{i=0}^{t-1} (t-i)y_i - R_t + tx_o + u_o .$$

Then it is easy to verify that an equivalent statement of the problem is to

(1')   minimize $\sum_{t=0}^{T-1} \bar{R}_t \, y_t$

subject to

(2') $u_t = \sum_{i=0}^{t} (t-i)y_i - R_t + tx_o + u_o \geq 0$

and $u_i, y_i \geq 0$ $\qquad\qquad i = 0 \ldots T .$

There is some further simplification possible. We can assume $x_o = 0$ and $u_o = 0$ for if they are not zero, then reduce each R_t by the amount $u_o + tx_o$. Thus, we can write the problem as follows.

(1")   Minimize $\sum_{t=0}^{T-1} \bar{R}_t y_t$

subject to

(2") $\sum_{i=0}^{t} (t-i)y_i = R_t + u_t ,$ $\qquad$ $t = 1 \ldots T$

where each $y_i, u_t \geq 0$.

3. DUALITY AND OPTIMALITY CONDITIONS

The dual of the latter system parallels closely the statement of the primal problem. This correspondence will be used to advantage in the graphical solution of the problem. These relations are exhibited as follows:

Primal problem	Dual problem
Minimize $\sum_{i=0}^{T-1} y_i \bar{R}_i$	Maximize $\sum_{t=1}^{T} \bar{y}_t R_t$
subject to	subject to
$\sum_{i=0}^{t} y_i(t-1) = R_t + u_t$	$\sum_{t=1}^{T} y_t(t-1) = \bar{R}_i - \bar{u}_i$
$y_i \geq 0$, $i = 0 \ldots, T-1$	$\bar{y}_t \geq 0$, $t = 1 \ldots, T$
$u_t \geq 0$ $t = 1 \ldots, T$	$\bar{u}_i \geq 0$ $i = 0 \ldots, T-1$
$(\bar{y}_t)$ = price associated with equations involving u_t	(y_t) = price associated with equations involving $\bar{u}_t$

Let a basic set of variables B be any set of T variables of the y_i's and u_i's whose determinant of coefficients is non-vanishing, it being understood that all other variables are set equal to zero. To every basic set of variables B satisfying the primal equations (not necessarily feasible) there is a complimentary basic set satisfying the dual equations (not necessarily feasible) with the properties:

Primal	Dual
If y_1 is in E, then	If $\bar{y}_t$ is in $\bar{E}$, then
$y_1\bar{u}_1 = 0$ for $i = 0,\ldots,T-1$.	$\bar{y}_t u_t = 0$ for $t = 1,\ldots,T$.
If u_t is in B, then $\bar{y}_1 = 0$ so	If $\bar{u}_t$ is in $\bar{B}$, then $y_t = 0$ so
that $u_1\bar{y}_1 = 0$, $i = 1,\ldots,T$	that $\bar{u}_1 y_1 = 0$, $i = 0,\ldots,T-1$.

If for any given basic set of primal variables E and corresponding basic set of dual variables $\bar{B}$ one can exhibit a feasible solution with these properties then it is optimal.

These dual relations have a useful graphical interpretation. Draw the requirement schedule $R = (R_1)$ as a histogram, and the dual $(\bar{R}_1)$. Let $\bar{R}$ be the lower convex subset of the points $(\bar{R}_1)$. The production plot $X = (X_1)$, a broken line connecting the series of points (X_1) lies above or just touches R at each time i. The complementary dual "production" plot $\bar{X}$ lies below or just touches $\bar{R}$. Both X and $\bar{X}$ will be convex broken line graphs since y_1 and $\bar{y}_t \geq 0$. We see that $y_t > 0$ (X has a break at time t) only if $\bar{X}$ is touching $\bar{R}$. Thus, y_t may be positive only if $\bar{R}_t$ is on $\bar{R}$. In addition, X has a break at t, $\bar{y}_t > 0$, only if X touches R at time t. u_t corresponds to the vertical distance from R up to X at time t, and $\bar{u}_t$ corresponds to the vertical distance from $\bar{R}$ down to $\bar{X}$ at time t.

These graphical concepts can be exhibited by the following simple example.

Let $R_0 = 0$, $R_1 = 1$, $R_2 = 3$, $R_3 = 7$, $R_4 = 14$; $u_1 = 1$, $c_1 = 1$, all 1; $e_0 = 4$, $e_1 = 2$, $e_2 = 3$, $e_3 = 1$.

Then

$$\overline{R}_0 = e_0 + c_1 + c_2 + c_3 + c_4 + d_2 + 2d_3 + 3d_4 = 14$$

$$\overline{R}_1 = c_1 + c_2 + c_3 + c_4 + d_3 + 2c_4 = 8$$

$$\overline{R}_2 = e_2 + c_3 + c_4 + d_4 = 6$$

$$\overline{R}_3 = e_3 + c_4 = 2$$

$$\overline{R}_4 = 0$$

Plot R and $\overline{R}$ as in Fig. 1. Complementary and dual solutions are indicated. Since the values of u_i, y_i, $\overline{u}_i$, $\overline{y}_i$ (see parenthetical figures beside variables in detached coefficient array below figure 1) are non-negative, this pair of solutions is optimal.

Fig. 1

Primal Problem

Primal → Dual ↓	(1) y_0	(2) y_1	y_2	(4) y_3	(1) u_1	u_2	u_3	u_4	1	
$\bar{y}_1$ (3)	1			−1					R_1 (1)	
$\bar{y}_2$	2	1			−1				R_2 (3)	
$\bar{y}_3$ (1)	3	2	1			−1			R_3 (7)	
$\bar{y}_4$ (2)	4	3	2	1			−1		R_4 (14)	
1	$\bar{R}_0$	$\bar{R}_1$	$\bar{R}_2$	$\bar{R}_3$						Max
	(14)	(8)	(6)	(2)						Min

Dual Problem

Dual → Primal ↓	(3) y_1	y_2	(1) y_3	(2) y_4	u_0	$\bar{u}_1$	(1) $\bar{u}_2$	$\bar{u}_3$	1	
y_0 (1)	1	2	3	4	1				R_0 (14)	
y_1 (2)		1	2	3		1			R_1 (8)	
y_2			1	2			1		R_2 (6)	
y_3 (4)				1				1	R_3 (2)	
1	R_1	R_2	R_3	R_4						Min
	(1)	(3)	(7)	(14)						Max

$\delta_j \geq 0$ $\bar{u}_0 \ \bar{u}_1 \ \bar{u}_2 \ \bar{u}_3 \ \bar{y}_1 \ \bar{y}_2 \ \bar{y}_3 \ \bar{y}_4$

(0)(0)(1)(0)(3)(0)(1)(2)

$\delta_j \leq 0$ $-u_1 \ -u_2 \ -u_3 \ -u_4 \ y_0 \ y_1 \ y_2 \ y_3$

(0)(−1)(0)(0)(−1)(−2)(0)(−4)

4. A SPECIAL COMPUTATION PROPERTY

Theorem 1: Given a basic set of variables we can find their values directly by successively solving one equation for one unknown at each step.

We show that since the values of the variables are uniquely determined, (the uniqueness follows from the assumption that their determinant of coefficients is non-vanishing) their values can be determined in the manner stated. For the purpose of this proof it will be convenient to solve for the x_1's. Consider the subset of those equations of the form $R_t = \sum_{i=1}^{t} x_1 + u_t$ where $u_t = 0$. From each of these equations subtract the preceding one in the subset. Then we have

$$
(3) \qquad R_a - R_b = x_a + x_{a+1} \cdots + x_{b+1}
$$
$$
R_c - R_b = x_b + \ldots + x_{c+1}, \text{etc.}
$$

Consider next the relations $x_{t+1} - x_t = y_t$ where $y_t = 0$. These imply that some of the successive x's will be equal. Write their sums in (3) as multiples of the x_1 with the smallest subscript. There will be at most one x in common between two successive equations; otherwise the equations are completely independent of one another. For example, if $y_b = 0$, then $x_{b+1} = x_b$. Otherwise, if $y_b > 0$, there is no x in common and the problem splits up into subproblems, each of which is solved separately.

It should be noted that this set of equations must be uniquely solvable for x_1 because the equations omitted from consideration are those in which u_t or y_t were not set equal to zero. These occur in only one equation each and are determined by the values of x_1.

There will be at most three different x's in any equation of (3). For example, $R_b - R_j = px_1 + qx_4 + rx_5$ where q and r may be zero. If there were four or more different x's, those not on either end could not have been uniquely determined a contradiction to our assumption. In the case where there are three different x's in an equation, the first one must be uniquely determined from the preceding equations and the last one from the equations following; otherwise, the value of the middle variable would not be unique. Then again in this case the problem splits up into subproblems.

Finally we are left with the case where there are two or less x's in equation of (3) and each equation has an x in common with the adjacent ones. Then, if all the equations had two x's there would be one more variable than there are equations. Thus, at least one equation has just one unknown which is solved directly and fed into the adjacent equations leaving just one unknown x to be solved in the next step, etc. Thus, the theorem is proved. Graphically, this means that a solution in x's can be constructed by ruler alone if one knows which u_i's are zero and which y_i's are positive.

5. CONVEX REQUIREMENT SCHEDULE

Theorem 2: If $r_t \geq r_{t-1}$, i.e., R is convex, then the optimal solution is given by the basic set of variables y_1 if $\bar{R}_1$ lies on $\bar{R}$, and u_1 if $\bar{R}_1$ is above $\bar{R}$ where $\bar{R}$ is the lower convex subset of points of $(\bar{R}_1)$.

The dual optimal solution $\bar{X}$ coincides with $\bar{R}$ and the construction of the primal optimal solution follows immediately.

The convexity of R assures the feasibility of this solution. Furthermore, the optimality conditions of #2 hold since

$\bar{R}_1$ on $\bar{R}$ implies $\bar{u}_1 = 0$, $\bar{y}_1$ in $\bar{\mathcal{B}}$, $u_1 = 0$, y_1 in $\mathcal{B}$.

$\bar{R}_1$ not on $\bar{R}$ implies $\bar{u}_1$ in $\bar{\mathcal{B}}$, $\bar{y}_1 = 0$, u_1 in $\mathcal{B}$, $y_1 = 0$.

Thus, $u_1\bar{y}_1 = 0$, $\bar{u}_1 y_1 = 0$ for $1 = 0,\ldots,T$.

These relations are exhibited by example of #3; see Fig. 1. Since R is convex to obtain the optimum solution given there, first plot $\bar{X}_t = \bar{R}$ the lower convex hull of $\bar{R}_t$. To obtain X_t note that $X_t = R_t$ wherever X_t bends (i.e., $t = 1, 3$); allow bends in X_t only where $\bar{u}_t = 0$ (i.e., $t = 1, 3$).

This means that in the convex requirement case, the two curves always bend and touch at the same values of t.

6. GENERAL REQUIREMENT SCHEDULE

This case has not been solved directly, but a rapid iterative procedure has been devised which in some cases (including the convex case of #5) gives the optimal solution directly in one step. By this procedure the $\overline{X}$ solution is always feasible. If, in addition, the X solution is feasible, it is also optimal. If some y_i or u_i's are negative then the largest one indicates the corresponding $\overline{u}_i$ or $\overline{y}_i$ to enter $\overline{B}$. This new variable is allowed to increase (graphically) until some other dual variable is driven out of $\overline{B}$, and a corresponding primal variable to enter B. These shifts in $\overline{B}$ and B follow from the price relations between primal and dual variables discussed in #3.

The procedure will be illustrated by Example 7 and Figures 2 and 3. The algebraic steps and their corresponding geometric construction are listed below.

Step 1. Let
$$\max_{t \leq T} \frac{R_t}{t} = \frac{R_a}{a}$$

Interpretation: Rotate a ruler about the point $R_o = 0$ downward until it touches R at R_a. In case of a tie, always choose the point with the smallest subscript.

Choose $\overline{X}_a = 0$, $\overline{y}_a$ in $\overline{B}$, and $u_a = 0$ not in B.

Step 2. Let
$$\overline{y}_a = \min_{\substack{t < a \\ \overline{R}_t \text{ on } \overline{R}}} \frac{\overline{R}_t}{a-t} = \frac{\overline{R}_q}{a-q} \ .$$

Interpretation: Rotate ruler upward about the point $X_a = 0$ until it touches $\bar{R}$ at $\bar{R}_{\bar{a}}$. Draw a line through these two points and extend it to the left.

Choose $y_{\bar{a}}$ in B and $\bar{u}_{\bar{a}} = 0$ not in $\bar{B}$.

Step 3. Let $\displaystyle \max_{t \leq \bar{a}} \frac{R_t}{t} = \frac{R_b}{b}$.

Interpretation: Rotate ruler as in Step 1 about $R_0 = 0$ until it touches R at $b \leq \bar{a}$. Then $X_b = \bar{y}_a (a - b)$. This is found graphically by extending the line in Step 2 to $t = b$.

Choose $\bar{y}_b$ in $\bar{B}$ and $u_b = 0$ not in B.

Step 4. Let $\bar{y}_{\bar{b}} = \displaystyle \min_{\substack{t < b \\ R_t \text{ on } \bar{R}}} \frac{R_t - \bar{X}_b}{b - t} = \frac{R_{\bar{b}} - X_b}{b - \bar{b}}$.

Interpretation: Rotate the ruler upward about X_b until it touches $\bar{R}$ at $\bar{R}_{\bar{b}}$. Draw a line through these points and extend it to the left.

Choose $y_{\bar{b}}$ in B and $\bar{u}_{\bar{b}} = 0$ not in $\bar{B}$.

Repeat these steps until $t = 0$ is reached. The X solution has been constructed in this process. The X solution is now drawn in Fig. 2 from the knowledge of which y's are in B and which u_1's are not in B. This can be done directly (cf. #3). At this stage the critical y_1's and u_1's alternate as t runs through 0 to T.

In the example we see that although no y_i happen to be negative, some u_i are < 0, with the largest violation being u_{22}. From the simplex algorithm, this indicates we should allow $\bar{y}_{22}$ to enter $\bar{E}$. We increase $\bar{y}_{22} > 0$ (by allowing a bend at $t = 22$ in $\bar{X}$) until some variable in $\bar{B}$ drops to zero, in this case $\bar{y}_{18}$. This indicates that u_{18} may enter B in place of u_{22}. The new solutions $\bar{X}$ and X are drawn in Fig. 3. Since X has no $u_i < 0$, or $y_i < 0$, it is the optimal solution to the problem. This illustrates that the starting procedure gives a solution which is usually quite close to the optimum.

EXAMPLE 2, FIG.2
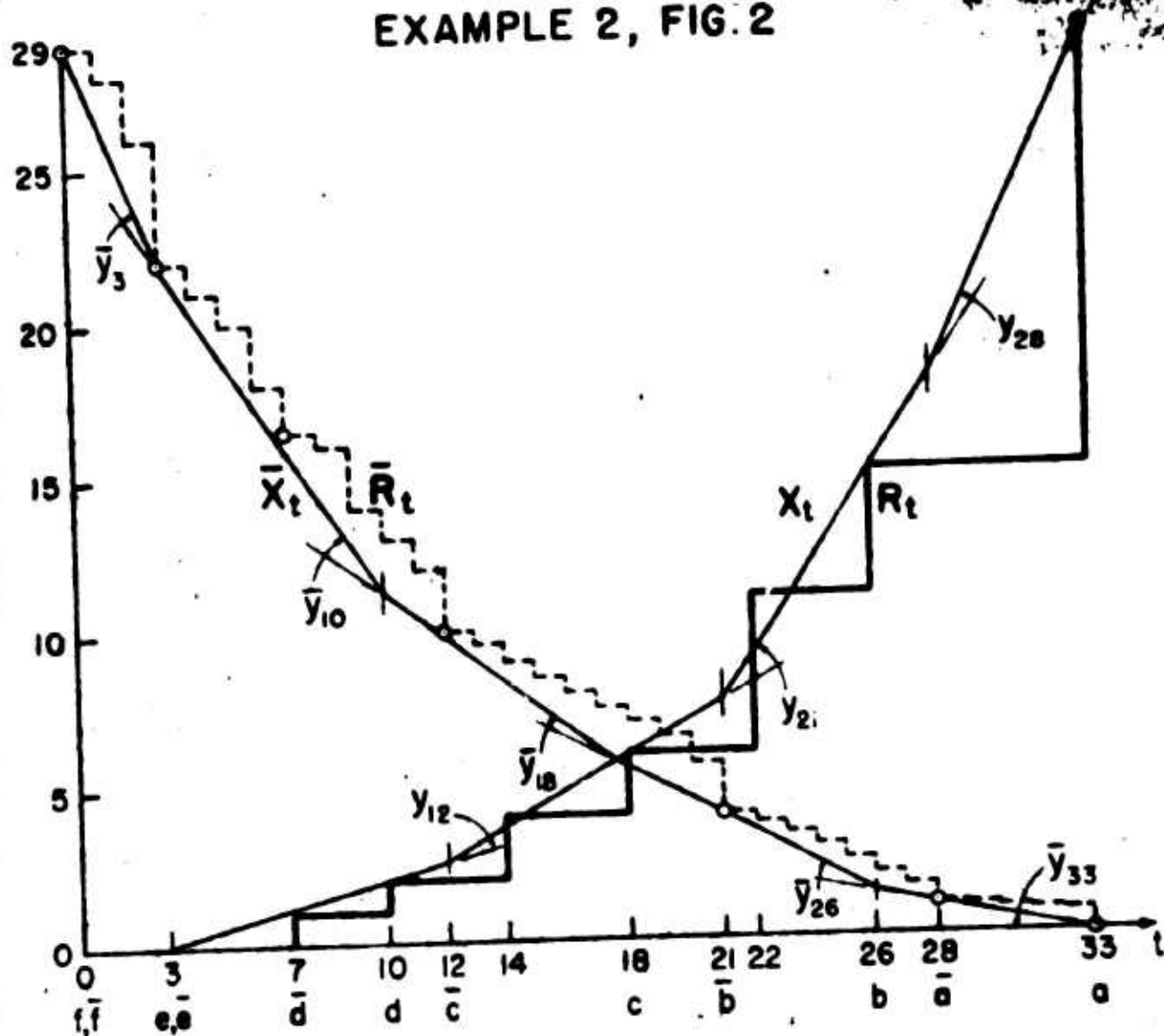
EXAMPLE 2, FIG.3
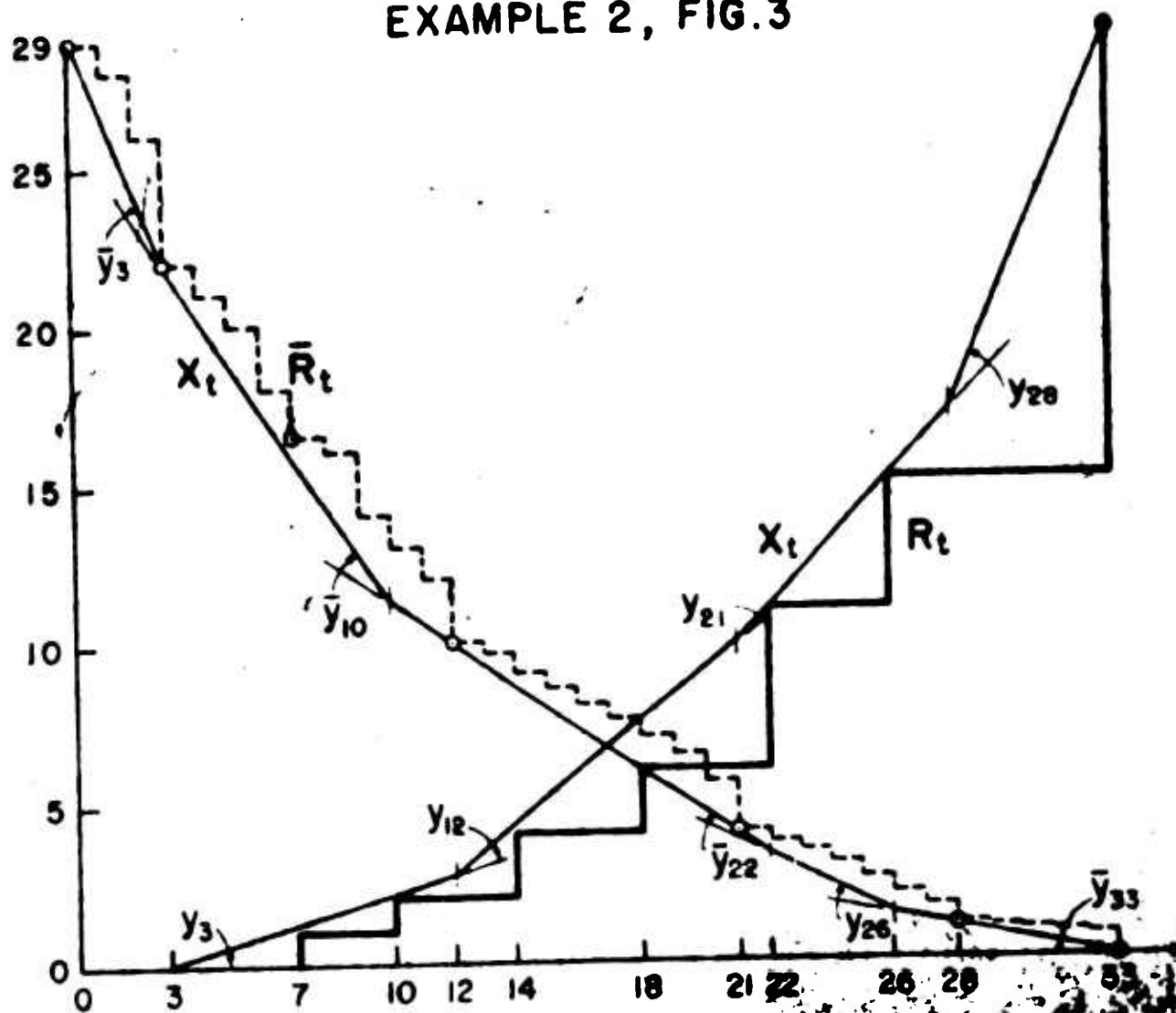
7. THE GENERAL CASE WITH DECREASING PRODUCTION ALLOWED

This problem is similar to that studied by Hoffman and Jacobs.
A decrease in production has an associated cost as a function of time.
In addition to the notation of #2 we have

$$
-y_t^* = \begin{cases} x_{t+1} - x_t & \text{if } x_{t+1} < x_t \\ 0 & \text{if } x_{t+1} \geq x_t \end{cases}
$$

where $y_t^* \geq 0$ is the magnitude of the decrease in production rate at time t,

e_t^* = the cost of decreasing the production rate one unit at time t (e_t^* is a given function of t).

Then similar to the definition of $\bar{R}_t$, see #2 , we define the potential total gain from decreasing the production one unit at time i,

$$
R_i^* = - e_i^* + \sum_{t=i+1}^{T} e_t + \sum_{t=1}^{T-1} (t-1)d_t
$$

for $i = 0, \ldots, T-1$ and $R_T^* = 0$.

The duality between R and $\bar{R}$ breaks down somewhat since we now have two dual functions $\bar{R}_i$ and $\bar{R}_i^*$ to plot; whereas in #2 we had just $\bar{R}_i$.

The problem can be stated as follows:

Minimize $\qquad \sum (\bar{R}_1 y_1 - \bar{R}_1^* y_1^*)$

where $\qquad \sum_{1=0}^{t} (t-1)(y_1 + y_1^*) \geq R_t + u_t$,

and $\qquad x_t = \sum_{1=0}^{t} (y_1 - y_1^*) \geq 0$

for $t = 0,\ldots,T$, and each y_1, y_1^*, $u_1 \geq 0$.

The corresponding matrix format for $T = 4$ is given below where the conditions $x_t \geq 0$ have been omitted.

	y_0	y_0^*	u_1	y_1	y_1^*	u_2	y_2	y_2^*	u_3	y_3	y_3^*	u_4	
$\bar{y}_1$	1	-1	-1										R_1
$\bar{y}_2$	2	-2		1	-1	-1							R_2
$\bar{y}_3$	3	-3		2	-2		1	-1	-1				R_3
$\bar{y}_4$	4	-4		3	-3		2	-2		1	-1	-1	R_4

Min 1 $\bar{R}_1$ $-\bar{R}_1^*$ $\bar{R}_2$ $-\bar{R}_2^*$ $\bar{R}_3$ $-\bar{R}_3^*$ $\bar{R}_4$ $-\bar{R}_4^*$

$\theta_j \geq 0$ $\quad \bar{u}_1^0$ $\bar{u}_1^0$ $\bar{y}_1$ $\bar{u}_2^0$ $\bar{u}_2^0$ $\bar{y}_2$ $\bar{u}_3^0$ $\bar{u}_3^0$ $\bar{y}_3$ $\bar{u}_2^0$ $\bar{u}_2^0$ $\bar{y}_4$

We plot $\bar{R}_1$ and $\bar{R}_1^*$ noting that $\bar{R}_1 \geq \bar{R}_1^*$ since $\bar{R}_1 - \bar{R}_1^* = e_1 + e_1^* \geq 0$. As in #2 we construct the dual $\bar{X}$ schedule but in this case it must lie between the two plots $\bar{R}_1$ and $\bar{R}_1^*$, i.e., $\bar{R}_1^* \leq \bar{X}_1 \leq \bar{R}_1$. Then $\bar{u}_1$ equals the distance from $\bar{R}_1$ down to $\bar{X}_1$ and $\bar{u}_1^*$ equals the distance from $\bar{X}_1$ down to $\bar{R}_1^*$.

Then $\bar{X}_1 = \bar{R}_1$, i.e., $\bar{u}_1^* = 0$, when $y_1 > 0$ is in ,

$\bar{X}_1 = \bar{R}_1^*$, i.e., $\bar{u}_1 = 0$, when $y_1^* > 0$ is in ,

in addition to the optimality conditions of #3.

Since the production rate is allowed to decrease we must check that $x_1 \geq 0$ for each i. It is recommended, however, that these conditions be ignored while working through the construction, thereby obtaining a tentative optimal solution which in fact it is if the X plot is non-decreasing.

Example 3 illustrates the usual case where no $x_1 < 0$ appeared in the tentative optimal solution. Hence, it is a true optimal. In Example 4 one of the x_1's is negative at the tentative solution and must be modified with an adjustment in #4. Figuratively, we add the condition that $x_1 = \sum_{t=0}^{i-1} y_t - \sum_{t=0}^{i-1} y_t^* \geq 0$. The element in the price vector corresponding to this equation, say v_1, will be represented finally by a discontinuous iterative jump (saltus) in the V plot at time i. This feature is illustrated in Fig. 4.

The solution of Examples 3 and 4 are explicit, but in general the solution requires an equilibration. In Example 5, we add the complex

plot $\overline{X}_t$ lying between $\overline{R}_t^*$ and $\overline{R}_t$ and touching each when required.
Start with any tentative basic (but not necessarily feasible) solution
for example $X_1 = R_1$, or perhaps the solution given in #2. Determine
the biggest violations on feasibility in either the primal or dual
solutions and adjust the $\hat{S}$ and $\overline{E}$ as in #6, taking into account the
more complicated relations which occur here.

It is easy to see that the fundamental computation property of
the basis demonstrated earlier for increasing production curves holds
also for the more general case as well — in fact, as far as E is
concerned the proof is the same if one allows the variable y_1 there
to take on positive and negative values. For E the proof of com-
putation property is complicated by the presence of possible jumps
in the $\overline{X}$ curve. Graphically, the effect of dropping x_r from the
basis of the primal (where primal now includes the equation

$x_r = \sum_{i=0}^{r-1} y_1 - \sum_{i=0}^{r-1} y_1^*$) is to allow a decrease w_r in the values of $\overline{R}_i$

and $\overline{R}_i^*$ for **all** $i < r$; accordingly, one continues to decrease these
values until there is some violation in the feasibility of the dual
solution.

Example 3. No $x_1 < 0$ is in this optimal solution.

		1			1			0	1			
	y_0	y_0^*	u_1	y_1	y_1^*	u_2	y_2	y_2^*	u_3	y_3	y_3^*	u_4
R_1(1)	1	-1	-1										$\bar{y}_1$(3)
R_2(1)	2	-2		1	-1	-1							$\bar{y}_2$(2)
R_3(1)	3	-3		2	-2		1	-1	-1				$\bar{y}_3$(0)
R_4(2)	4	-4		3	-3		2	-2		1	-1	-1	$\bar{y}_4$(1)
Min	11	-9		7	-5		3	-1		1	0		
$-\delta$	0	-2	-3	-2	0	-2	-1	-1	0	0	-1	-1	

Example 4.

		1			1		2		2			
	y_0	y_0^*	u_1	y_1	y_1^*	u_2	y_2	y_2^*	u_3	y_3	y_3^*	u_4
R_1(1)	1	-1	-1										$\bar{y}_1$(1)
R_2(1)	2	-2		1	-1	-1							$\bar{y}_2$(0)
R_3(1)	3	-3		2	-2		1	-1	-1				$\bar{y}_3$(1)
R_4(2)	4	-4		3	-3		2	-2		1	-1	-1	$\bar{y}_4$(1)
Min	8	-6		7	-4		4	-3		1	0		-1
$-\delta$	0	-2	-1	-2	-1	0	-1	0	-1	0	-1	-1	

This primal solution is infeasible ($x_3 < 0$). Therefore, we add a condition $x_3 = \sum_{0}^{2} y_1 - \sum_{0}^{2} y_1^* \geq 0$.

Forcing $x_3 = 0$ is the same as dropping the dual points $\bar{R}_i + \bar{R}_i^*$ i = 0,1,2, down in Fig. 5 until some element in $\bar{B}$ drops to zero. In this case one can easily check that there is a tie. Both $\bar{y}_3$ and $\bar{u}_1^*$ vanish simultaneously. Either may be dropped from $\bar{B}$ and the two optimal primal solutions are given by Fig. 6 and Fig. 4.

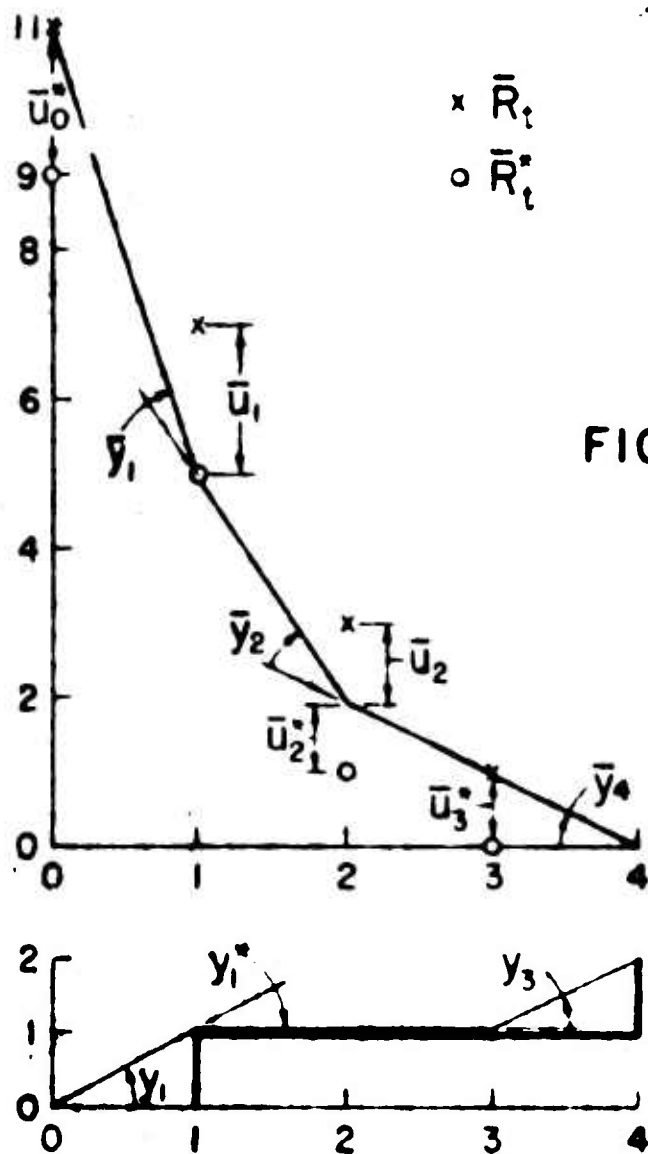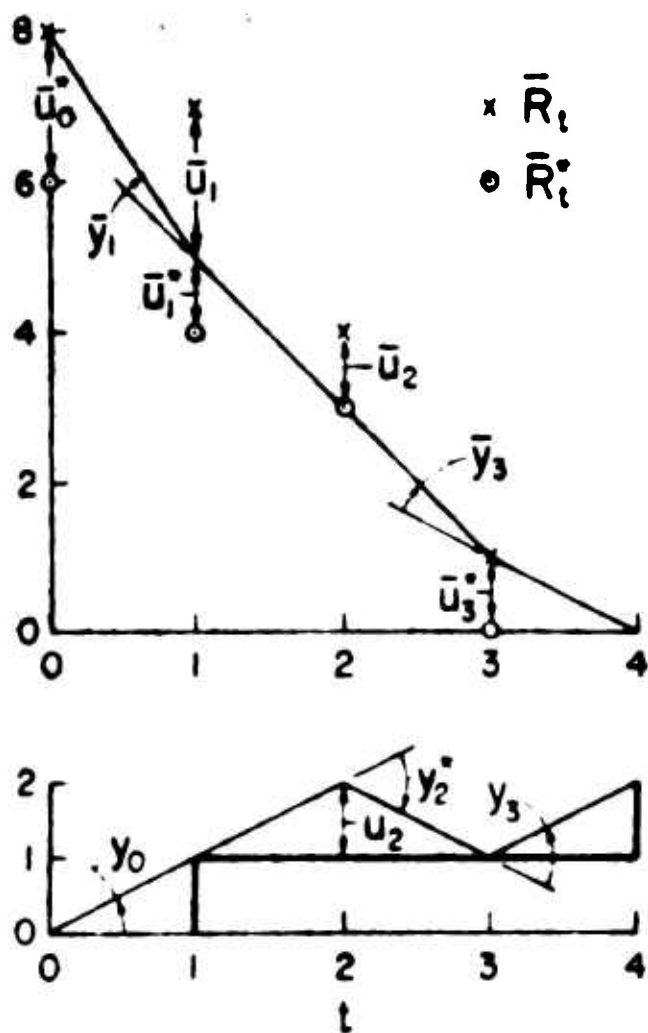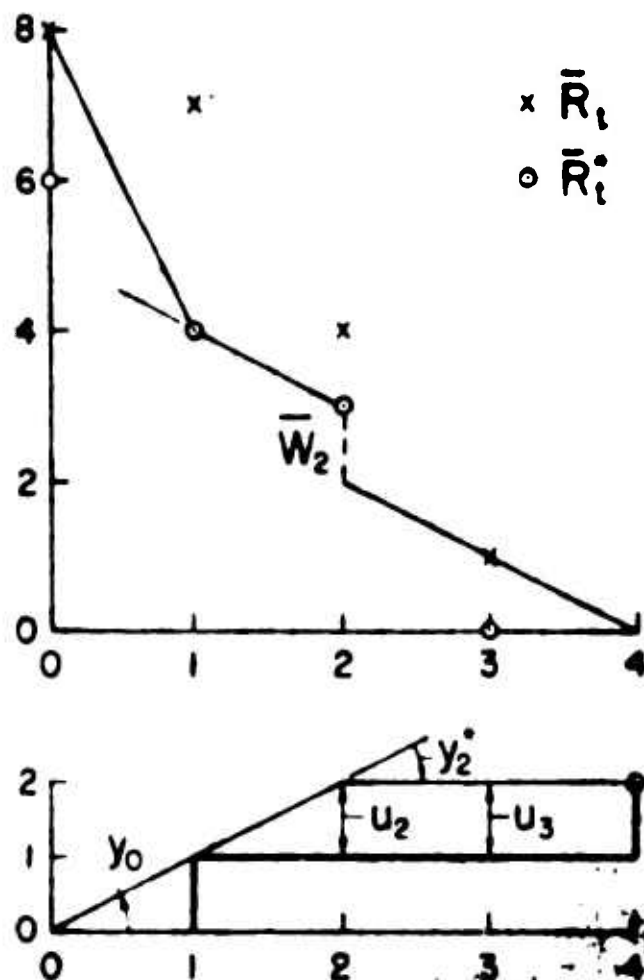|	1				1		1	1	0					
	y_0	y_0^*	u_1	y_1	y_1^*	u_2	y_2	y_2^*	u_3	y_3	y_3^*	u_4	x_3	
$R_1(1)$	1	-1	-1											$\bar{y}_1(3)$
$R_2(1)$	2	-2		1	1	-1								$\bar{y}_2(0)$
$R_3(1)$	3	-3		2	-2		1	-1	-1					$\bar{y}_3(0)$
$R_4(2)$	4	-4		3	-3		2	-2		1	1	-1		$\bar{y}_4(1)$
0	1	-1		1	-1		1	-1					-1	$\bar{u}_2(1)$
Min	8	-6		7	-4		4	-3		1	0		-1	
-6	0	-2	-3	-3	0	0	-1	0	0	0	-1	-1	-1	

FIG. 4

FIG. 5

FIG. 6

It is interesting to verify the results of Hoffman and Jacobs [2] by our methods. Some of the arguments are close to theirs but apply to more general cases. In [2] they consider the special case: $c_i = 0$, $d_i = 1$, $e_i = \lambda$, $e_i^* = 0$. Thus in our notation we have

$$\bar{R}_i = \lambda + \sum_{t=1}^{T-1} (t-i) \quad \text{and} \quad \bar{R}_i^* = \sum_{t=1}^{T-1} (t-i) \quad \text{and the } \bar{R}^* \text{ and } \bar{R} \text{ plots are}$$

strictly convex with $\bar{R}_t - \bar{R}_t^* = \lambda$ for each t.

We first state

Theorem 3: If $\bar{R}^*$ is strictly convex, then $y_t^* > 0$, $u_t > 0$ for any t is impossible for an optimal solution.

Proof: If $y_t^* > 0$, $u_t > 0$, then $\bar{u}_t^* = 0$, $\bar{y}_t = 0$. Thus $\bar{X}_{t-1} < \bar{R}_{t-1}^*$ by the convexity of $\bar{R}^*$, which is impossible unless $w_{t-1} > 0$ and $x_t = 0$. But $y_t^* > 0$ implies that $x_{t+1} < x_t = 0$, a contradiction.

This result includes Theorem 1 of [2], i.e., for a fixed λ, if $x_{t+1} < x_t$, then $X_t = R_t$.

Theorem 2 of [2] states that the final $u_t = 0$. This would always follow when there is no cost for decreasing production for the last period, that is, $e_{T-1}^* = 0$.

Theorem 3 of [2] states that $R_t \le X_t \le K_t$ where K is the upper convex envelope of R. For assume for some first t that $X_t > K_t$, then $u_t > 0$ and by our Theorem 3 $y_t^* = 0$ or $y_t \ge 0$ so that X is convex with steeper slope at t than K which has a non-increasing slope with time. The two curves X_t and K_t can never meet for $t = T$ contradicting $u_T = 0$ (Theorem 2 of [2]).

In #4 and #5 of [2] the special case of convex R is discussed. It is easily checked that our Theorem 3 implies that X must be convex

... of X ... ... 0 ... ... Theorem 3.
$q_t = 0$. ... ... $h_{t+} > X_{t+}$

... ... ed to

... to discuss ... ...

... ...

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[2] Hoffman, A. J., W. Jacobs, Smooth Patterns of Production, Management Science, Vol. 1, No. 1, October, 1954, p. 86.